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Selcuk, O.; Talman, A.J.J.

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GAMES WITH GENERAL COALITIONAL STRUCTURE

By

Özer Selçuk, Dolf Talman

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Tilburg University

Games with general coalitional structure

Özer Selçuk* **Dolf Talman†**

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Abstract

This paper introduces a new solution concept for cooperative games with general coalitional structure in which only certain sets of players, including the set of all players, are able to form feasible coalitions. The solution concept takes into account the marginal contribution of players. This marginal contribution can be a joint contribution of several players and is equally divided among those players. Any set system representing a coalitional structure induces a collection of coalitional trees, whose nodes may consist of subsets of players. As solution we take the average of the marginal contribution vectors that correspond to all coalitional trees. The solution is efficient and several other properties are studied and some special cases are considered.

Keywords: TU game, cooperation structure, marginal contribution, set system, Shapley value

JEL Classification Number: C71

1 Introduction

A situation in which a finite set of agents or players can cooperate, may form coalitions of players and the total payoff obtained by this cooperation, its worth, is freely distributed among the coalition members is called a cooperative game with transferable utility or simply TU game. Assuming that the grand coalition of all players forms, an efficient solution concept deals with the distribution of the total payoff created through this cooperation. The Shapley value is the most well known solution concept for TU games. The Shapley value is the average of all marginal (contribution) vectors in the game. Each marginal vector in the game corresponds to a permutation on the set of players. At such a vector, each player receives as payoff the difference of the worth of the coalition consisting of himself together with all his successors in the permutation and the worth of the coalition consisting of only his successors.

Unlike the standard cooperative game theory assumption stating that all coalition of players may form, in many practical situations the collection of feasible coalitions in which players are able to cooperate and obtain some worth is restricted by some social, economical, hierarchical or technical structure. In the literature,

*O. Selcuk, CentER, Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: o.selcuk@tilburguniversity.edu.

†A.J.J. Talman, CentER, Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: talman@tilburguniversity.edu.

the incomplete nature of communication structure to cooperate is often represented by a graph on the set of players. In TU games whose communication structure is represented with a graph, so-called *graph games*, only the members of a connected set of players are assumed to be able to cooperate. The best known solution concept for graph games is the Myerson value introduced by Myerson (1977). The Myerson value is defined as the Shapley value of the so called Myerson restricted game. Although graph games are more general than TU games and can be applied to many situations, they may still be not enough to explain some real life phenomenon. To illustrate this insufficiency, consider a cooperative game defined on a set of political parties. Being members of a large spectrum of political space, let us focus on one extreme left-wing party (party L), one extreme right-wing party (party R) and one center party (party C). Since party L and party C are able to cooperate in order to form a government, then if we try to represent this situation with a graph these two parties need to be connected. Similarly, party R and party C need to be connected in the graph because they might be able to cooperate as well. Also the three parties together may form a feasible coalition and form a national government. However, each of the three parties are typically not be able to form the government by themselves, because they are too small. A single party is therefore not a feasible coalition in this case. This situation cannot be represented by a graph structure because by definition the single nodes of a graph are connected sets.

In this paper, we allow for an arbitrary collection of feasible coalitions that are able to cooperate and obtain some worth. We only assume that the grand coalition is always able to form. Considering the example on political parties again, one could assume only the existence of feasible coalitions of party L with party C, of party R with party C, and the grand coalition of all three parties. For such coalitional games, as solution concept we propose the average coalitional tree solution, being the average of the marginal vectors that correspond to all maximal nested sets in the set system representing the coalitional structure. A maximal nested set is a collection of feasible coalitions such that for any two different coalitions in the collection, either one of them is a subset of the other or they are disjoint, and, moreover, the union of two or more disjoint coalitions in the collection is not a feasible coalition. Depending on the coalitional structure, a maximal nested set may consist of less coalitions than the number of players. In the example of political parties above, there are two maximal nested sets, each containing the grand coalition and one of the two other feasible two-party coalitions. In this example there are no singletons in a maximal nested set, because the single parties are not feasible. To each maximal nested set a coalitional tree corresponds. In a coalitional tree each node is a coalition, which may not be feasible by its own, but together with its set of successors it is always feasible and it is a member of the maximal nested set. For each of these coalitional trees, a marginal vector is defined, at which the players at a node receive together as payoff the marginal contribution when they join to their successors in the tree and this payoff is equally distributed among them. In the example above, if the maximal nested set consisting of the grand coalition and the coalition of party L together with party C is considered, then at the corresponding marginal vector party L and party C equally share their own joint worth and party R receives its marginal contribution to them. The average coalitional tree solution is single-valued and satisfies efficiency, linearity, equal treatment of equivalent players, and the null player property, amongst others. Also some special cases are considered.

Aguilera et al. (2010) also consider games with arbitrary coalitional structures and define what they call the Shapley value for such games. Instead of maximal nested sets they consider maximal chains in the set system and take as solution the average of the marginal vectors corresponding to all marginal chains. A chain is a collection of feasible coalitions such that for any two different members one of them is a subset of the other. Hence, chains do not consider the cases where one or more players are able to join simultaneously to two or more feasible disjoint coalitions whose union is not feasible, to form a larger feasible coalition. A chain only considers the marginal contribution of a set of players when it joins only one feasible coalition. Therefore, the two concepts of maximal chain and maximal nested set differ from each other. A maximal chain is always a nested set, not necessarily being maximal, and when a maximal chain is not a maximal nested set it must be a proper subset of at least one maximal nested set. A similar approach with maximal chains is employed in Grabisch and Lange (2009) for more restricted structures. They consider regular set systems where the feasible coalitions form a poset whose maximal chains all have the same length. They propose an axiomatization of the Shapley value for this class of games.

In the literature, a large collection of papers consider specific classes of set systems as a way to represent limited cooperation among the players. In all of these researches, some restrictions are assumed on the set systems. Among these researches, Algaba et al. (2001) consider union stable cooperation structures, Bilbao and Edelman (2000) consider convex geometries, Bilbao et al. (2001) consider matroids, Algaba et al. (2003) consider antimatroids, Bilbao and Ordonez (2007) consider augmenting systems, Ui et al. (2011) consider complete coalition structures, and Koshevoy and Talman (2011) consider building sets. For building sets the solution given in Koshevoy and Talman (2011) coincides with the average coalitional tree solution.

The structure of this paper is as follows. Basic definitions and notation are given in Section 2. Section 3 introduces the new solution concept for coalitional games. Some properties of the solution concept are given in Section 4. Section 5 considers some special coalitional structures.

2 Preliminaries

2.1 Coalitional game

A transferable utility (TU) game with coalitional structure or *coalitional game* is a triple $(N, v, \mathcal{F})$, where $N = \{1, ..., n\}$ is a finite set of players, $\mathcal{F} \subseteq 2^N$ is a *set system* on N representing the coalitional structure and containing N, and $v : \mathcal{F} \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. A set $S \in \mathcal{F}$ is a *feasible coalition* and the real number $v(S)$ represents the *worth* of feasible coalition S, which can be freely distributed among its members. We denote the set of coalitional games with a fixed player set N by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears we write $(v, \mathcal{F})$ instead of $(N, v, \mathcal{F})$ when we refer to a coalitional game with player set N and whose coalitional structure is represented by $\mathcal{F}$ and characteristic function by v.

A *payoff vector* is a vector $x \in \mathbb{R}^N$ with ith component x_i the payoff to player $i \in N$. A *value* on $\mathcal{G}_N$ is a function $\xi \colon \mathcal{G}_N \to \mathbb{R}^N$ that assigns to coalitional game

$(v, \mathcal{F}) \in \mathcal{G}_N$ the payoff vector $\xi(v, \mathcal{F}) \in \mathbb{R}^N$. In the sequel, we use notation $x(S) = \sum_{i \in S} x_i$ for any payoff vector $x \in \mathbb{R}^N$ and $S \subseteq N$. $|A|$ denotes the cardinality of a finite set A.

2.2 Coalitional tree

Given a partition $\mathcal{P}$ of the set of players N, $(\mathcal{P}, T)$ stands for a *coalitional directed graph*, or *coalitional digraph*, on $\mathcal{P}$, where $T \subseteq \{(P_1, P_2) \mid P_1, P_2 \in \mathcal{P}, P_1 \neq P_2\}$ is a collection of directed links on the set of members of the partition $\mathcal{P}$ of N. A coalitional digraph may be seen as a generalization of a *directed graph*, or *digraph*, where the nodes, being elements of N, are replaced with subsets of N that together form a partition of N and the directed links are defined on these subsets. Given a partition $\mathcal{P}$ of N and a coalitional digraph $(\mathcal{P}, T)$ on $\mathcal{P}$, a sequence of different members of $\mathcal{P}$, $(P_1, ..., P_k)$ with $k \geq 2$, is a *directed coalitional path* in $(\mathcal{P}, T)$, from P_1 to P_k if $(P_h, P_{h+1}) \in T$ for $h = 1, ..., k-1$. If there exists a directed coalitional path in $(\mathcal{P}, T)$ from $P \in \mathcal{P}$ to $P' \in \mathcal{P}$, then P' is a *successor* of P and P is a *predecessor* of P'. If $(P, P') \in T$ then P' is an *immediate successor* of P and P is an *immediate predecessor* of P'. For any $P \in \mathcal{P}$, $S_T(P)$ denotes the union of all successors of P in the coalitional digraph $(\mathcal{P}, T)$ and $\overline{S}_T(P)$ denotes the union of successors of P together with the members of P, i.e., $\overline{S}_T(P) = S_T(P) \cup P$. Furthermore, $\mathcal{I}_T(P)$ denotes the set of immediate successors of P in $(\mathcal{P}, T)$, i.e., $\mathcal{I}_T(P) = \{P' \in \mathcal{P} | (P, P') \in T\}$. For simplicity of notation and if no ambiguity appears we write T when we refer to a coalitional diagraph $(\mathcal{P}, T)$ on a given partition $\mathcal{P}$ of N.

A coalitional digraph $(\mathcal{P}, T)$ on a given partition $\mathcal{P}$ of N is a *coalitional tree* if there exists a unique member of the partition $\mathcal{P}$, called the *coalitional root* of T and denoted by $r(T)$, having no predecessors in T and there is a unique directed coalitional path in T from $r(T)$ to every other member of the partition.

3 Average coalitional tree solution

For a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$, we assume that the grand coalition N is always a feasible coalition and is therefore an element of the coalitional structure $\mathcal{F}$. There are no other restrictions imposed on the set system $\mathcal{F}$ of feasible coalitions. The idea is that the grand coalition N will form and the problem is how to distribute its worth $v(N)$ among the agents. As solution concept, we propose to take the average of the marginal contribution vectors induced by all maximal nested sets of the set system representing the coalitional structure. Nested sets of a set system are introduced by Postnikov (2005).

Definition 3.1 Given a coalitional structure $\mathcal{F}$ on N, a subset $\mathcal{X}$ of $\mathcal{F}$ is a *nested set* of $\mathcal{F}$ if it satisfies the following conditions:
(i) For any two different $X_1, X_2 \in \mathcal{X}$ it holds that either $X_1 \subset X_2$ or $X_2 \subset X_1$ or $X_1 \cap X_2 = \emptyset$;
(ii) For any collection of h, $h \geq 2$, disjoint nonempty subsets $X_1,\ldots,X_h$ in $\mathcal{X}$ it holds that $X_1 \cup \cdots \cup X_h \notin \mathcal{F}$;
(iii) $N \in \mathcal{X}$.

A nested set of a coalitional structure is a collection of feasible coalitions, including the set of all players, such that for any two different members either one of them is a subset of the other one or they are disjoint, and, moreover, the union of two or more disjoint members is not a feasible coalition. Notice that every chain of a coalitional structure $\mathcal{F}$ is a nested set of $\mathcal{F}$, where $\mathcal{Y}$ is a *chain* of $\mathcal{F}$ if $N \in \mathcal{Y}$ and for any two different $Y_1, Y_2 \in \mathcal{Y}$ it holds that either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$.

A nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ is *maximal* if there does not exist any other nested set $\mathcal{X}'$ of $\mathcal{F}$ that contains $\mathcal{X}$. A maximal nested set defines a unique way to build the grand coalition by letting one or simultaneously several players join to one or more feasible coalitions to form bigger feasible coalitions, starting from disjoint minimal (by set inclusion) coalitions in the set system. Notice that a maximal chain is not necessarily a maximal nested set and that a maximal nested set may not be a maximal chain. If a maximal chain is not a maximal nested set, then there exists a maximal nested set that contains the chain as a proper subset.

Example 3.1 Consider the coalitional structure $\mathcal{F} = \{\{1\}, \{1,2\}, \{2,3\}, \{1,2,3,4\}\}$ on $N = \{1,2,3,4\}$. This set system has two maximal nested sets, $\mathcal{X}^1 = \{\{1\}, \{1,2\}, \{1,2,3,4\}\}$ and $\mathcal{X}^2 = \{\{1\}, \{2,3\}, \{1,2,3,4\}\}$. In $\mathcal{X}^1$, player 2 joins feasible singleton player 1 to form feasible coalition $\{1,2\}$ and players 3 and 4 join simultaneously the latter coalition to form the grand coalition. In $\mathcal{X}^2$, player 4 joins to both feasible singleton player 1 and minimal feasible coalition $\{2,3\}$ to form immediately the grand coalition. The two feasible coalitions $\{1,2\}$ and $\{2,3\}$ cannot be members of a same maximal nested set because one is not a subset of the other and their intersection is nonempty. On the other hand, the two disjoint feasible coalitions $\{1\}$ and $\{2,3\}$ can be members of the same maximal nested set, because their union, $\{1,2,3\}$, is not a feasible coalition. The maximal nested set $\mathcal{X}^1$ is also a maximal chain, but $\mathcal{X}^2$ is not. On the other hand, $\{\{2,3\}, \{1,2,3,4\}\}$ is a maximal chain that is not a maximal nested set, because it is a proper subset of $\mathcal{X}^1$.

Any coalitional structure on N contains at least one maximal nested set. To see this, note that N itself is a nested set, which does not need to be maximal. If it is not a maximal nested set, then we can include any other feasible coalition. If this new collection of two feasible coalitions is again not maximal, we continue with including feasible coalitions which do not violate the definition of a nested set, and so on. Because the number of feasible coalitions is finite, at some point we end up with a maximal nested set. This argument also shows that every feasible coalition is member of at least one maximal nested set.

For a coalitional structure $\mathcal{F}$ on N, $\overline{\mathcal{X}}^{\mathcal{F}}$ denotes the collection of maximal nested sets of $\mathcal{F}$. Notice that in case the coalitional structure $\mathcal{F}$ contains all subsets of N, i.e., $\mathcal{F} = 2^N$, then the number of maximal nested sets is maximal and equal to $n!$ and all maximal nested sets are maximal chains.

Given a coalitional structure $\mathcal{F}$ on N, for a nested set $\mathcal{X}$ of $\mathcal{F}$ and $i \in N$, the set $M_{\mathcal{X}}(i)$ denotes the unique minimal element of $\mathcal{X}$ that contains player i. Notice that due to conditions (i) and (iii) of Definition 3.1 this set is well defined.

Lemma 3.1 *For any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ of a coalitional structure $\mathcal{F}$ on N, it holds that for every $X \in \mathcal{X}$ there exists $i \in N$ such that $M_{\mathcal{X}}(i) = X$.*

Proof. Suppose there exists a feasible coalition $X \in \mathcal{X}$ for which there is no $i \in N$ with $M_{\mathcal{X}}(i) = X$. Since $\mathcal{X}$ is a maximal nested set and $M_{\mathcal{X}}(i) \neq X$ for all $i \in X$, it holds that $M_{\mathcal{X}}(i) \subset X$, $M_{\mathcal{X}}(i) \in \mathcal{X}$ and $i \in M_{\mathcal{X}}(i)$ for all $i \in X$. Since $M_{\mathcal{X}}(i) \in \mathcal{X}$ for all $i \in X$, there exist $i_1, \ldots, i_k \in X$ such that $M_{\mathcal{X}}(i_1), \ldots, M_{\mathcal{X}}(i_k)$ are disjoint and $\cup_{h=1}^{k} M_{\mathcal{X}}(i_h) = X$. Since X is a feasible coalition this violates condition (ii) of Definition 3.1. ∎

Example 3.2 Consider the coalitional structure $\mathcal{F} = \{\{1,2\},\{3\},\{2,3,4\},\{1,2,3,4,5\}\}$ on $\{1,2,3,4,5\}$. It has two maximal nested sets, $\mathcal{X}^1 = \{\{1,2\},\{3\},\{1,2,3,4,5\}\}$ and $\mathcal{X}^2 = \{\{3\},\{2,3,4\},\{1,2,3,4,5\}\}$. For these maximal nested sets we have $M_{\mathcal{X}^1}(1) = M_{\mathcal{X}^1}(2) = \{1,2\}$, $M_{\mathcal{X}^1}(3) = \{3\}$, $M_{\mathcal{X}^1}(4) = M_{\mathcal{X}^1}(5) = \{1,2,3,4,5\}$, and $M_{\mathcal{X}^2}(1) = M_{\mathcal{X}^2}(5) = \{1,2,3,4,5\}$, $M_{\mathcal{X}^2}(2) = M_{\mathcal{X}^2}(4) = \{2,3,4\}$, $M_{\mathcal{X}^2}(3) = \{3\}$.

In the example we see that, for two distinct players the minimal elements of a maximal nested set that contain these players can be the same.

Definition 3.2 For a collection $\mathcal{S}$ of subsets of N, two players $i, j \in N$ are *equivalent* with respect to $\mathcal{S}$ if $\{S \in \mathcal{S} \mid i \in S\} = \{S \in \mathcal{S} \mid j \in S\}$.

For a collection $\mathcal{S}$ of subsets of N and $i \in N$, $P_{\mathcal{S}}(i)$ denotes the set of equivalent players of i with respect to $\mathcal{S}$. Note that $i \in P_{\mathcal{S}}(i)$ holds for all $i \in N$, which means that each player is equivalent to himself. Moreover, if $\{i\} \in \mathcal{S}$ for some $i \in N$, then there exists no other player which is equivalent to player i, and so $P_{\mathcal{S}}(i) = \{i\}$.

Remark 3.1 Given a coalitional structure $\mathcal{F}$ on N, for a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, two players $i, j \in N$ are equivalent with respect to $\mathcal{X}$ if and only if $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$.

Remark 3.2 For a coalitional structure $\mathcal{F}$ on N, if two players $i, j \in N$ are equivalent with respect to $\mathcal{F}$, then i and j are equivalent with respect to every maximal nested set in $\overline{\mathcal{X}}^{\mathcal{F}}$.

Remark 3.1 follows from condition (i) of Definition 3.1. Remark 3.2 is an immediate result of the fact that each maximal nested set is a subset of the set system representing the coalitional structure.

A maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ of a coalitional structure $\mathcal{F}$ on N induces a partition $\mathcal{P}^{\mathcal{X}}$ of N into sets of equivalent players with respect to $\mathcal{X}$, with $P_{\mathcal{X}}(i)$ the partition member containing the equivalent players of player $i \in N$. For $P \in \mathcal{P}^{\mathcal{X}}$, since P is a set of equivalent players with respect to $\mathcal{X}$, it holds that $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$ for all $i, j \in P$. We denote $M_{\mathcal{X}}(P)$ to be the set $M_{\mathcal{X}}(i)$ for any $i \in P$, $P \in \mathcal{P}^{\mathcal{X}}$.

Given a maximal nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ on N and the corresponding partition $\mathcal{P}^{\mathcal{X}}$, the coalitional digraph $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is given by $(P_1, P_2) \in T^{\mathcal{X}}$ if $M_{\mathcal{X}}(P_1) \supset M_{\mathcal{X}}(P_2)$ and there exists no $X \in \mathcal{X}$ with $M_{\mathcal{X}}(P_1) \supset X \supset M_{\mathcal{X}}(P_2)$. The next theorem shows that this coalitional digraph is a coalitional tree.

Theorem 3.1 *For any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ of a coalitional structure $\mathcal{F}$ on N, the coalitional digraph $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree satisfying the following properties:*
(i) Its coalitional root $r(T^{\mathcal{X}})$ is equal to $\{i \in N \mid M_{\mathcal{X}}(i) = N\}$;
(ii) For any $P \in \mathcal{P}^{\mathcal{X}}$ it holds that $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$;
(iii) For any $P \in \mathcal{P}^{\mathcal{X}}$ it holds that $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid (P, P') \in T^{\mathcal{X}}\}$ is the unique maximal partition of $S_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P) \setminus P$ into elements of $\mathcal{X}$.

Proof. To show that $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree we need to prove the uniqueness of a coalitional root and the uniqueness of a directed coalitional path in the tree from the coalitional root to any other member of $\mathcal{P}^{\mathcal{X}}$. By Lemma 3.1 the set $R = \{i \in N \mid M_{\mathcal{X}}(i) = N\}$ is nonempty and consists of equivalent players with respect to $\mathcal{X}$. Hence, $R \in \mathcal{P}^{\mathcal{X}}$. Since there exists no $P \in P^{\mathcal{X}}$ with $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(R) = N$, it holds that R has no predecessor in $T^{\mathcal{X}}$. Since $\mathcal{X}$ is a maximal nested set there exists a unique directed path in $T^{\mathcal{X}}$ from R to any other member of $\mathcal{P}^{\mathcal{X}}$. This implies that $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree with the coalitional root being the set R, which also proves property (i).

Property (ii) is shown by induction. Take any $P \in \mathcal{P}^{\mathcal{X}}$ without successor in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$, then $\overline{S}_{T^{\mathcal{X}}}(P) = P$ and $P \subseteq M_{\mathcal{X}}(P)$. Suppose there exists $i \in M_{\mathcal{X}}(P) \setminus P$, then i is not equivalent to the players in P with respect to $\mathcal{X}$ and therefore $M_{\mathcal{X}}(i) \subset M_{\mathcal{X}}(P)$. This implies the existence of a directed coalitional path in $T^{\mathcal{X}}$ from P to $P_{\mathcal{X}}(i)$, which contradicts that P has no successors. Next, we show $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$ if $\overline{S}_{T^{\mathcal{X}}}(P') = M_{\mathcal{X}}(P')$ for all P' satisfying $(P, P') \in T^{\mathcal{X}}$. Let $P_1 \ldots, P_k$ be the collection of immediate successors of P in $T^{\mathcal{X}}$, then $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(P_i) = \overline{S}_{T^{\mathcal{X}}}(P_i)$ for all $i = 1, \ldots, k$. Since $\overline{S}_{T^{\mathcal{X}}}(P) = (\cup_{i=1}^{i=k} \overline{S}_{T^{\mathcal{X}}}(P_i)) \cup P$ and $P \subseteq M_{\mathcal{X}}(P)$, this implies $M_{\mathcal{X}}(P) \supseteq \overline{S}_{T^{\mathcal{X}}}(P)$. Suppose $j \in M_{\mathcal{X}}(P) \setminus \overline{S}_{T^{\mathcal{X}}}(P)$. Since j is not equivalent to the players in P with respect to $\mathcal{X}$ and $j \in M_{\mathcal{X}}(P)$, we have $M_{\mathcal{X}}(j) \subset M_{\mathcal{X}}(P)$. This implies the existence of a directed coalitional path in $T^{\mathcal{X}}$ from P to $P_{\mathcal{X}}(j)$, which contradicts that $j \notin \overline{S}_{T^{\mathcal{X}}}(P)$.

To show property (iii) let $P_1, \ldots, P_k$ be the collection of immediate successors of $P \in \mathcal{P}^{\mathcal{X}}$ in $T^{\mathcal{X}}$. Condition (ii) implies that $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$ and $\overline{S}_{T^{\mathcal{X}}}(P_j) = M_{\mathcal{X}}(P_j)$ for all $j = 1 \ldots, k$. Hence, $M_{\mathcal{X}}(P) \setminus P = \overline{S}_{T^{\mathcal{X}}}(P) \setminus P = S_{T^{\mathcal{X}}}(P) = \cup_{j=1}^{k} \overline{S}_{T^{\mathcal{X}}}(P_j) = \cup_{j=1}^{k} M_{\mathcal{X}}(P_j)$. Since $\mathcal{X}$ is a maximal nested set, $M_{\mathcal{X}}(P_j) \cap M_{\mathcal{X}}(P_h) = \emptyset$ for all $j \neq h$, and $M_{\mathcal{X}}(P_j) \in \mathcal{X}$ for $j = 1, \ldots, k$, this implies that $\{\overline{S}_{T^{\mathcal{X}}}(P_1), \ldots, \overline{S}_{T^{\mathcal{X}}}(P_k)\}$ is the unique maximal partition of $S_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P) \setminus P$ into members of $\mathcal{X}$. ∎

A coalitional tree on a set of players N that is induced by a maximal nested set may be seen as a generalization of a tree on N in which the nodes of the tree are coalitions of equivalent players with respect to the maximal nested set instead of individual players.

Since each maximal nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ on N contains the grand coalition N and by applying Lemma 3.1, there exist players for which the grand coalition is the minimal set in $\mathcal{X}$ containing them. According to property (i) the root of the coalitional tree $T^{\mathcal{X}}$ induced by $\mathcal{X}$ precisely consists of these players. So, if a maximal nested set is considered as a way to build the grand coalition, the root of the induced coalitional tree is the final set of equivalent players that simultaneously join after all other sets of equivalent players have joined. According to property (ii), the players at a node of $T^{\mathcal{X}}$ together with the players in all succeeding nodes is the minimal set in $\mathcal{X}$ that contains any player at that node. A direct implication of this property is that the union of the successors of any partition member $P \in \mathcal{P}^{\mathcal{X}}$ together with the elements of P is a member of the set system $\mathcal{F}$, and hence is a feasible coalition. Property (iii) says that for each member X of a maximal nested set, it holds that if we delete from X all players for which X is the minimal set containing them, then there is a unique maximal partition of the set of remaining players in X into members of the maximal nested set.

Example 3.3 Consider the coalitional structures $\mathcal{F} = \{\{1\},\{2,3\},\{1,2,3\}\}$ and $\mathcal{F}' = \{\{1\},\{2\},\{2,3\},\{1,2,3\}\}$ on $N = \{1,2,3\}$. Both $\mathcal{F}$ and $\mathcal{F}'$ contain two maximal nested sets, $\mathcal{X}^1 = \{\{1\},\{1,2,3\}\}$ and $\mathcal{X}^2 = \{\{2,3\},\{1,2,3\}\}$ for $\mathcal{F}$, and $\mathcal{Y}^1 = \{\{1\},\{2\},\{1,2,3\}\}$ and $\mathcal{Y}^2 = \{\{2\},\{2,3\},\{1,2,3\}\}$ for $\mathcal{F}'$. For the partitions induced by these maximal nested sets we have $\mathcal{P}^{\mathcal{X}^1} = \{\{1\},\{2,3\}\}$, $\mathcal{P}^{\mathcal{X}^2} = \{\{2,3\},\{1\}\}$ and $\mathcal{P}^{\mathcal{Y}^1} = \{\{1\},\{2\},\{3\}\}$, $\mathcal{P}^{\mathcal{Y}^2} = \{\{2\},\{3\},\{1\}\}$. The coalitional trees on these partitions are equal to $T^{\mathcal{X}^1} = \{(\{2,3\},\{1\})\}$, $T^{\mathcal{X}^2} = \{(\{1\},\{2,3\})\}$ and $T^{\mathcal{Y}^1} = \{(\{3\},\{1\}),(\{3\},\{2\})\}$, $T^{\mathcal{Y}^2} = \{(\{1\},\{3\}),(\{3\}, \{2\})\}$. Note that $M_{\mathcal{X}^1}(\{1\}) = \{1\}$ and $M_{\mathcal{X}^1}(\{2,3\}) = \{1,2,3\}$. Since $M_{\mathcal{X}^1}(\{1\}) \subset M_{\mathcal{X}^1}(\{2,3\})$ we have $(\{2,3\},\{1\}) \in T^{\mathcal{X}^1}$. Similarly for $\mathcal{X}^2$, we have $M_{\mathcal{X}^2}(\{1\}) = \{1,2,3\}$ and $M_{\mathcal{X}^2}(\{2,3\}) = \{2,3\}$ which means $M_{\mathcal{X}^2}(\{2,3\}) \subset M_{\mathcal{X}^2}(\{1\})$. Hence, $(\{1\},\{2,3\}) \in T^{\mathcal{X}^2}$. The graphical representation of these coalitional trees is depicted in Figure 1. The root of $T^{\mathcal{X}^1}$ is $\{2,3\}$ with succeeding set $S_{T^{\mathcal{X}^1}}(\{2,3\}) = \{1\}$, whereas $\{1\}$ is the root of $T^{\mathcal{X}^2}$ with succeeding set $S_{T^{\mathcal{X}^2}}(\{1\}) = \{2,3\}$. In $T^{\mathcal{Y}^1}$ the set $\{3\}$ is the root and the sets $\overline{S}_{T^{\mathcal{Y}^1}}(\{1\}) = \{1\}$ and $\overline{S}_{T^{\mathcal{Y}^1}}(\{2\}) = \{2\}$ partition the set $S_{T^{\mathcal{Y}^1}}(\{3\}) = \{1,2\}$ of players in the succeeding sets of the root into members of $\mathcal{Y}^1$. Coalitional tree $T^{\mathcal{Y}^2}$ has $\{1\}$ as root and feasible coalition $S_{T^{\mathcal{Y}^2}}(\{1\}) = \{2,3\}$ is the only succeeding set of the root. Since both $\{2\}$ and $\{2,3\}$ are feasible coalitions in $\mathcal{F}'$ but $\{3\}$ is not, only $\{3\}$ can be an immediate successor of $\{1\}$ and therefore $S_{T^{\mathcal{Y}^2}}(\{3\}) = \{2\}$.

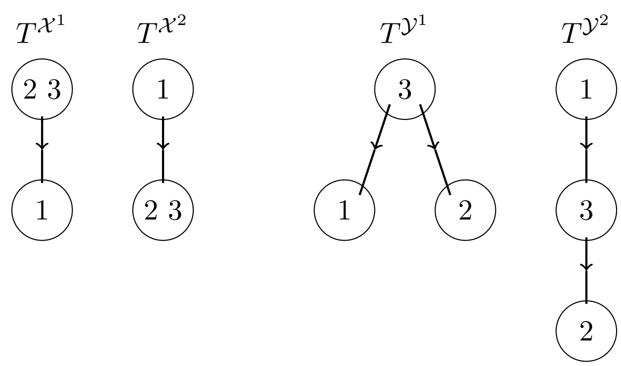


Figure 1: The coalitional trees of $\mathcal{F}$ and $\mathcal{F}'$ in Example 3.3.

The following example shows that the sets of equivalent players may differ in different coalitional trees of the same set system.

Example 3.4 Consider the coalitional structure $\mathcal{F} = \{\{3\},\{8\},\{1,2\},\{1,8\},\{2,3,4,5\},\{1,2,3,4,5\},N\}$ on $N = \{1,2,3,4,5,6,7,8\}$. $\mathcal{F}$ contains two maximal nested sets, $\mathcal{X}^1 = \{\{1,2\},\{3\},\{8\},\{1,2,3,4,5\},\{1,2,3,4,5,6,7,8\}\}$ and $\mathcal{X}^2 = \{\{3\},\{1,8\},\{2,3,4,5\},\{1,2,3,4,5,6,7,8\}\}$. For the maximal nested set $\mathcal{X}^1$, we have $M_{\mathcal{X}^1}(1) = M_{\mathcal{X}^1}(2) = \{1,2\}$, $M_{\mathcal{X}^1}(3) = \{3\}$, $M_{\mathcal{X}^1}(4) = M_{\mathcal{X}^1}(5) = \{1,2,3,4,5\}$, $M_{\mathcal{X}^1}(6) = M_{\mathcal{X}^1}(7) = \{1,2,3,4,5,6,7,8\}$, $M_{\mathcal{X}^1}(8) = \{8\}$, and for the maximal nested set $\mathcal{X}^2$, we have $M_{\mathcal{X}^2}(1) = M_{\mathcal{X}^2}(8) = \{1,8\}$, $M_{\mathcal{X}^2}(2) = M_{\mathcal{X}^2}(4) = M_{\mathcal{X}^2}(5) = \{2,3,4,5\}$, $M_{\mathcal{X}^2}(3) = \{3\}$, $M_{\mathcal{X}^2}(6) = M_{\mathcal{X}^2}(7) = \{1,2,3,4,5,6,7,8\}$. In $\mathcal{X}^1$, player 1 is equivalent to player 2, player 4 is equivalent to player 5, and player 6 is equivalent to player 7. However, in $\mathcal{X}^2$, player 1 is equivalent to player 8, player 6 is equivalent to player 7, and players 2, 4, 5 are equivalent to each other. For the induced partitions of equivalent players we have $\mathcal{P}^{\mathcal{X}^1} = \{\{1,2\},\{3\},\{8\},\{4,5\},\{6,7\}\}$ and $\mathcal{P}^{\mathcal{X}^2} = \{\{1,8\},\{3\},\{2,4,5\},\{6,7\}\}$. The graphical representation of the corresponding coalitional trees $T^{\mathcal{X}^1}$ and $T^{\mathcal{X}^2}$ is depicted in Figure 2. Coalition $S_{T^{\mathcal{X}^1}}(\{6,7\}) = \{1,2,3,4,5,8\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^1}}(\{4,5\}) = \{1,2,3,4,5\}$ and $\overline{S}_{T^{\mathcal{X}^1}}(\{8\}) = \{8\}$, and $S_{T^{\mathcal{X}^1}}(\{4,5\}) = \{1,2,3\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^1}}(\{1,2\}) = \{1,2\}$ and $\overline{S}_{T^{\mathcal{X}^1}}(\{3\}) = \{3\}$. Similarly, coalition $S_{T^{\mathcal{X}^2}}(\{6,7\}) = \{1,2,3,4,5,8\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^2}}(\{2,4,5\}) = \{2,3,4,5\}$ and $\overline{S}_{T^{\mathcal{X}^2}}(\{1,8\}) = \{1,8\}$, and $S_{T^{\mathcal{X}^2}}(\{2,4,5\}) = \overline{S}_{T^{\mathcal{X}^2}}(\{3\}) = \{3\}$.

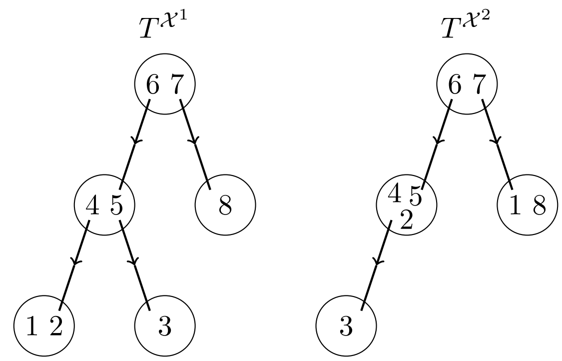


Figure 2: The coalitional trees of $\mathcal{F}$ in Example 3.4.

Given a coalitional game $(v,\mathcal{F}) \in \mathcal{G}_N$ on the set of players N, we define for every maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ the marginal vector $m^{\mathcal{X}}(v,\mathcal{F})$ as the payoff vector given by

$$m_i^{\mathcal{X}}(v,\mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(v(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} v(M_{\mathcal{X}}(P))\Big),\ i \in N.$$

For a marginal vector corresponding to the coalitional tree induced by a maximal nested set, each set of equivalent players receives as total payoff the marginal contribution when these players simultaneously join to the players in the sets of successors in the tree. The total payoff available for a set of equivalent players is distributed equally among the players of the set. The intuition behind this marginal vector is as follows. Given a maximal nested set and the corresponding coalitional tree, each set of equivalent players is the smallest set that can join to its set of successors to form a bigger feasible coalition. So, each set of equivalent players should receive its marginal contribution when joining to its successors. On the individual level, since all members of a set of equivalent players join simultaneously, this marginal contribution is equally divided among them. Every maximal nested set of the underlying coalitional structure of the game is as likely to occur. This gives the following definition of a solution.

Definition 3.3 For a coalitional game $(v,\mathcal{F}) \in \mathcal{G}_N$, the *average coalitional tree solution* is the payoff vector $ACT(v,\mathcal{F})$ given by

$$ACT(v,\mathcal{F}) = \frac{1}{|\overline{\mathcal{X}}^{\mathcal{F}}|}\sum_{\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}} m^{\mathcal{X}}(v,\mathcal{F}).$$

The average coalitional tree solution of a coalitional game is the average of the marginal vectors that corrspond to the coalitional trees induced by all maximal nested sets of the coalitional structure. Like the Shapley value, the solution considers for each player his marginal contributions. However, for some players it may not be possible to join a feasible coalition individually in order to form a larger feasible coalition. To illustrate such a case, consider the coalitional structure $\mathcal{F}$ given in Example 3.3. For this coalitional structure, there exists no feasible coalition to which player 2 can join, make it a larger feasible coalition and receive his own marginal contribution. However, together with player 3, player 2 is able to join to the feasible coalition consisting of singleton player 1. This contribution is realized in the coalitional tree $T^{\mathcal{X}^1}$ of the example. The joint marginal contribution of players 2 and 3 while joining to singleton player 1 is divided equally among the two players.

Example 3.5 Consider a coalitional game $(v, \mathcal{F})$ where the coalitional structure $\mathcal{F}$ is the one given in Example 3.4. For the characteristic function $v : 2^N \to \mathbb{R}$, let $v(S) = |S|^2$ for all $S \in 2^N$. The two maximal nested sets, $\mathcal{X}^1$ and $\mathcal{X}^2$, induce the two marginal vectors $m^{\mathcal{X}^1} = m^{\mathcal{X}^1}(v, \mathcal{F})$ and $m^{\mathcal{X}^2} = m^{\mathcal{X}^2}(v, \mathcal{F})$. For $m^{\mathcal{X}^1}$ it holds that $m_1^{\mathcal{X}^1} = m_2^{\mathcal{X}^1} = (v(\{1, 2\}) - v(\emptyset))/2 = v(\{1, 2\})/2 = 2$, $m_3^{\mathcal{X}^1} = v(\{3\}) - v(\emptyset) = 1$, $m_4^{\mathcal{X}^1} = m_5^{\mathcal{X}^1} = (v(\{1, 2, 3, 4, 5\}) - v(\{1, 2\}) - v(\{3\}))/2 = 10$, $m_6^{\mathcal{X}^1} = m_7^{\mathcal{X}^1} = (v(N) - v(\{1, 2, 3, 4, 5\}) - v(\{8\}))/2 = 19$, $m_8^{\mathcal{X}^1} = v(\{8\}) - v(\emptyset) = 1$, and so $m^{\mathcal{X}^1} = (2, 2, 1, 10, 10, 19, 19, 1)^\top$. Similarly, for $m^{\mathcal{X}^2} = m^{\mathcal{X}^2}(v, \mathcal{F})$ it holds that $m_1^{\mathcal{X}^2} = m_8^{\mathcal{X}^2} = (v(\{1, 8\}) - v(\emptyset))/2 = 2$, $m_2^{\mathcal{X}^2} = m_4^{\mathcal{X}^2} = m_5^{\mathcal{X}^2} = (v(\{2, 3, 4, 5\}) - v(\{3\}))/3 = 5$, $m_3^{\mathcal{X}^2} = v(\{3\}) - v(\emptyset) = 1$, $m_6^{\mathcal{X}^2} = m_7^{\mathcal{X}^2} = (v(N) - v(\{2, 3, 4, 5\}) - v(\{1, 8\}))/2 = 22$, and so $m^{\mathcal{X}^2} = (2, 5, 1, 5, 5, 22, 22, 2)^\top$. The average coalitional tree solution of the game is the average of these two marginal vectors, so $ACT(v, \mathcal{F}) = \frac{1}{2}(m^{\mathcal{X}^1} + m^{\mathcal{X}^2}) = (2, 7/2, 1, 15/2, 15/2, 41/2, 41/2, 3/2)^\top$.

We remark that the average coalitional tree solution is defined for cooperative games with completely arbitrary coalitional structure. The only assumption being made is that the grand coalition can be formed. In Section 5 special cases of coalitional structures are considered.

4 Properties of the average coalitional tree solution

In this section we provide some properties that are satisfied by the average coalitional tree solution. Let $\mathcal{G}_N$ be again the set of coalitional games on the player set $N = \{1, \ldots, n\}$. Recall that we assume the grand coalition N is always a feasible coalition.

4.1 Efficiency

For a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$, a payoff vector $x \in \mathbb{R}^N$ is *efficient* if x distributes the worth $v(N)$ of the grand coalition, i.e. $x(N) = v(N)$. A value ξ on $\mathcal{G}_N$ is *efficient* if for any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ the payoff vector $\xi(v, \mathcal{F})$ is efficient.

Proposition 4.1 *The average coalitional tree solution is efficient.*

Proof. For a given coalitional game $(v, \mathcal{F})$ in $\mathcal{G}_N$, the average coalitional tree solution is equal to the average of the marginal vectors corresponding to the coalitional trees induced by all maximal nested sets of $\mathcal{F}$. Since each marginal vector distributes the worth $v(N)$ of N over all players, the efficiency of the average coalitional tree solution follows. ∎

4.2 Linearity

A value ξ on $\mathcal{G}_N$ satisfies *linearity* if for any two coalitional games $(v, \mathcal{F})$ and $(w, \mathcal{F})$ in $\mathcal{G}_N$ it holds that for any $a, b \in \mathbb{R}$,

$$\xi(av + bw, \mathcal{F}) = a\xi(v, \mathcal{F}) + b\xi(w, \mathcal{F}),$$

where the characteristic function $av+bw$ is defined as $(av+bw)(S) = av(S)+bw(S)$ for all $S \in \mathcal{F}$.

Proposition 4.2 *The average coalitional tree solution satisfies linearity.*

Proof. Given two coalitional games $(v, \mathcal{F})$ and $(w, \mathcal{F})$ in $\mathcal{G}_N$ and real numbers $a, b \in \mathbb{R}$, since the coalitional structure is represented by $\mathcal{F}$ for $(v, \mathcal{F})$, $(w, \mathcal{F})$ and $(av + bw, \mathcal{F})$, the collection of maximal nested sets is the same for each of these three coalitional games. Since the ACT solution of a coalitional game is a linear combination of the marginal vectors induced by all maximal nested sets, the ACT solution satisfies linearity. ∎

4.3 Equal treatment of equivalent players

For a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$, two players $i, j \in N$ are *equivalent* if they are equivalent with respect to $\mathcal{F}$. A value ξ on the class of coalitional games satisfies the *equal treatment of equivalent players property* if for any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ it holds that $\xi_i(v, \mathcal{F}) = \xi_j(v, \mathcal{F})$ whenever $i \in N$ and $j \in N$ are equivalent players for the game $(v, \mathcal{F})$. The intuition behind this property is that since equivalent players for a coalitional game always join simultaneously to the same sets, these players should receive the payoff.

Proposition 4.3 *The average coalitional tree solution satisfies the equal treatment of equivalent players property.*

Proof. Take any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$, maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and two players $i, j \in N$ being for $(v, \mathcal{F})$. Let $(P^{\mathcal{X}}, T^{\mathcal{X}})$ be the coalitional tree induced by $\mathcal{X}$. According to Remark 3.2, players i and j are equivalent also with respect to $\mathcal{X}$. This implies the existence of $P \in \mathcal{P}^{\mathcal{X}}$ such that $i, j \in P$. Hence,

$$m_i^{\mathcal{X}}(v, \mathcal{F}) = m_j^{\mathcal{X}}(v, \mathcal{F}) = \frac{1}{|P|}\Big(v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P'))\Big).$$

Since

$$ACT(v, \mathcal{F}) = \frac{1}{|\overline{\mathcal{X}}^{\mathcal{F}}|} \sum_{\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}} m^{\mathcal{X}}(v, \mathcal{F}),$$

it implies that $ACT_i(v, \mathcal{F}) = ACT_j(v, \mathcal{F})$. ∎

4.4 Null player property

For TU games, since any subset of players is feasible, a player is able to join any subset of players he doesn't belong to. For such games, a null player is defined to be a player whose marginal contribution is zero when joining to any set of players. For our setting, since not all coalitions are feasible, a player may not be able to join all coalitions. Moreover, a player may need other players to be able to join to a feasible coalition and also we may have cases where players are not joining to a single feasible coalition but to several disjoint feasible coalitions.

Definition 4.1 Given a coalitional structure $\mathcal{F}$ on N and $S \in \mathcal{F}$, $\{S_1, \ldots, S_k\}$ is a *maximal subpartition* of S if it satisfies the following conditions:

(i) $S_h \subset S$ for all $h \in \{1, \ldots, k\}$ and $S \setminus (\bigcup_{h=1}^{k} S_h) \neq \emptyset$;

(ii) $S_h \in \mathcal{F}$ and $S_h \cap S_m = \emptyset$ holds for all distinct $h, m \in \{1, \ldots, k\}$;

(iii) $S' \cup (\bigcup_{m \in M} S_m) \notin \mathcal{F}$ for all $S' \subseteq (S \setminus \bigcup_{h=1}^{k} S_h)$ and $M \subseteq \{1, \ldots, k\}$.

For a feasible coalition $S \in \mathcal{F}$, a collection of disjoint feasible coalitions $S_1, \ldots S_k$ is a maximal subpartition of S if their union is a proper subset of S and it is not possible to find any other such collection that is obtained by combining some of the members of the collection with some other players in S. A maximal subpartition of a feasible coalition S does not need to be unique and is the empty set if there exists no $S' \in \mathcal{F}$ such that $S' \subset S$. For $S \in \mathcal{F}$, $\overline{\mathcal{D}}_{\mathcal{F}}(S)$ denotes the collection of maximal subpartitions of S. Note that if the empty set is a maximal subpartition of a feasible coalition S, then there exists no other maximal subpartitions of S.

Example 4.1 Consider the coalitional structure $\mathcal{F} = \{\{1\}, \{2\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}\}$ on $N = \{1, 2, 3, 4, 5\}$. For the grand coalition N there exist three maximal subpartitions, $\{\{1\}, \{2, 3\}\}$, $\{\{1\}, \{4, 5\}\}$, and $\{\{2, 3\}, \{4, 5\}\}$. Furthermore, $\overline{\mathcal{D}}_{\mathcal{F}}(\{2, 3\}) = \{\{\{2\}\}\}$ and $\overline{\mathcal{D}}_{\mathcal{F}}(\{4, 5\}) = \emptyset$.

Definition 4.2 For a coalitional game $(v, \mathcal{F})$, player $i \in N$ is a *null player* if $v(S) - \sum_{Q \in \mathcal{D}} v(Q) = 0$ for all $S \in \mathcal{F}$ and $\mathcal{D} \in \overline{\mathcal{D}}_{\mathcal{F}}(S)$ satisfying $i \in S \setminus \bigcup_{Q \in \mathcal{D}} Q$.

A player is a null player if the contribution is zero when he and possibly other players join to a maximal subpartition of a feasible coalition he belongs to. Notice that if a player $i \in N$ is a null player and $\{i\} \in \mathcal{F}$, then $v(\{i\}) = 0$.

A value ξ on the class of coalitional games satisfies the *null player property* if for any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ it holds that $\xi_i(v, \mathcal{F}) = 0$ whenever $i \in N$ is a null player for $(v, \mathcal{F})$.

Proposition 4.4 *The average coalitional tree solution satisfies the null player property.*

Proof. Take any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ and $i \in N$ such that player i is a null player for $(v, \mathcal{F})$. Consider any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and let $P_{\mathcal{X}}(i) = P$. By property (iii) of Theorem 3.1, $\{S_{T^{\mathcal{X}}}(P') \mid P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)\}$ is a maximal subpartition of $S_{T^{\mathcal{X}}}(P)$ not containing i. Since i is a null player for $(v, \mathcal{F})$ and $\overline{S}_{T^{\mathcal{X}}}(Q) = M_{\mathcal{X}}(Q)$ for any $Q \in \mathcal{P}^{\mathcal{X}}$, we have

$$v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P')) = v(\overline{S}_{T^{\mathcal{X}}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(\overline{S}_{T^{\mathcal{X}}}(P')) = 0.$$

This implies

$$m_i^{\mathcal{X}}(v, \mathcal{F}) = \frac{1}{|P|}\Big(v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P'))\Big) = 0.$$

Since the ACT solution is the average of the marginal vectors corresponding to the coalitional trees induced by all maximal nested sets of $\mathcal{F}$, it holds that $ACT_i(v, \mathcal{F}) = 0$. ∎

4.5 Independence of closed coalitions

If a feasible coalition of a coalitional structure is disjoint to any other feasible coalition that does not contain it or is not a subset of it and if no subset of it is able to join other feasible coalitions to form a larger feasible coalition, then this coalition is called a closed coalition.

Definition 4.3 Given a coalitional structure $\mathcal{F}$ on N, a feasible coalition $Q \in \mathcal{F}$ is a *closed coalition* if it satisfies the following conditions:
(i) For all $S \in \mathcal{F}$, $S \neq Q$, either $Q \subset S$ or $Q \supset S$ or $Q \cap S = \emptyset$;
(ii) For any $Q' \subseteq Q$ and $S_1, \ldots, S_k \in \mathcal{F}$ satisfying $S_h \cap Q = \emptyset$ for $h = 1, \ldots, k$ it holds that $(\bigcup_{j=1}^{k} S_j) \cup Q' \notin \mathcal{F}$.

Subsets of a closed coalition are not able to contribute to the coalitions that also contain players outside the coalition. Because of this property, the members of a closed coalition should receive together just the worth of that coalition, the payoffs of players inside the closed coalition should only depend on the worths of the feasible subcoalitions of the closed coalition, and the payoffs of players outside the closed coalition should be independent of these worths.

Definition 4.4 A value ξ on the class of coalitional games satisfies *independence of closed coalitions* if for any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ and closed coalition $Q \in \mathcal{F}$ the following conditions hold:
(i) $\sum_{i \in Q} \xi_i(v, \mathcal{F}) = v(Q)$;
(ii) For any coalitional game $(w, \mathcal{F}) \in \mathcal{G}_N$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \subseteq Q$, it holds that $\xi_i(v, \mathcal{F}) = \xi_i(w, \mathcal{F})$ for all $i \in Q$;
(iii) For any coalitional game $(w, \mathcal{F}) \in \mathcal{G}_N$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \supseteq Q$ or $S \cap Q = \emptyset$, it holds that $\xi_i(v, \mathcal{F}) = \xi_i(w, \mathcal{F})$ for all $i \in N \setminus Q$.

Proposition 4.5 *The average coalitional tree solution satisfies independence of closed coalitions.*

Proof. Take any coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ and closed coalition $Q \in \mathcal{F}$. We first show that $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Suppose there exists a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ such that $Q \notin \mathcal{X}$. Since Q is a closed coalition and $\mathcal{X} \subseteq \mathcal{F}$, conditions (i) and (ii) of Definition 4.3 imply that $\mathcal{X} \cup \{Q\}$ is a nested set, which contradicts that $\mathcal{X}$ is a maximal nested set.

To prove condition (i) of Definition 4.4 take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. By Lemma 3.1 and Theorem 3.1, there exists $P \in \mathcal{P}^{\mathcal{X}}$ such that $M_{\mathcal{X}}(P) = \overline{S}_{T^{\mathcal{X}}}(P) = Q$. This implies $\sum_{i \in Q} m_i^{\mathcal{X}}(v, \mathcal{F}) = v(M_{\mathcal{X}}(P)) = v(Q)$. Since the ACT solution is the average of the marginal vectors corresponding to the coalitional trees induced by all maximal nested sets, we have $\sum_{i \in Q} ACT_i(v, \mathcal{F}) = v(Q)$.

To show condition (ii) of Definition 4.4, take any $i \in Q$ and coalitional game $(w, \mathcal{F})$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \subseteq Q$. Both coalitional games $(v, \mathcal{F})$ and $(w, \mathcal{F})$ have the same coalitional structure $\mathcal{F}$ and $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, then

$$m_i^{\mathcal{X}}(v, \mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(v(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} v(M_{\mathcal{X}}(P))\Big)$$

and

$$m_i^{\mathcal{X}}(w, \mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(w(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} w(M_{\mathcal{X}}(P))\Big).$$

$Q \in \mathcal{X}$ and $i \in Q$ imply both $M_{\mathcal{X}}(i) \subseteq Q$ and $M_{\mathcal{X}}(P) \subseteq Q$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Since $v(S) = w(S)$ for all $S \subseteq Q$ we obtain $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Hence, $m_i^{\mathcal{X}}(v, \mathcal{F}) = m_i^{\mathcal{X}}(w, \mathcal{F})$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and therefore $ACT_i(v, \mathcal{F}) = ACT_i(w, \mathcal{F})$.

To show condition (iii) of Definition 4.4, take any $i \in N \setminus Q$ and coalitional game $(w, \mathcal{F}) \in \mathcal{G}_N$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \supseteq Q$ or $S \cap Q = \emptyset$. Again both coalitional games $(v, \mathcal{F})$ and $(w, \mathcal{F})$ have the same coalitional structure $\mathcal{F}$ and $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Since $Q \in \mathcal{X}$ is a closed coalition and $i \in N \setminus Q$, either $M_{\mathcal{X}}(i) \cap Q = \emptyset$ or $M_{\mathcal{X}}(i) \supset Q$. If $M_{\mathcal{X}}(i) \cap Q = \emptyset$ then $M_{\mathcal{X}}(P) \cap Q = \emptyset$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$, which implies $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and therefore $m_i^{\mathcal{X}}(v, \mathcal{F}) = m_i^{\mathcal{X}}(w, \mathcal{F})$. If $M_{\mathcal{X}}(i) \supset Q$, then $M_{\mathcal{X}}(P') \supseteq Q$ for precisely one $P' \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and $M_{\mathcal{X}}(P) \cap Q = \emptyset$ for all other $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Since $v(S) = w(S)$ for all $S \supseteq Q$ or $S \cap Q = \emptyset$, this implies $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and therefore $m_i^{\mathcal{X}}(v, \mathcal{F}) = m_i^{\mathcal{X}}(w, \mathcal{F})$. The two cases together imply that $ACT_i(v, \mathcal{F}) = ACT_i(w, \mathcal{F})$. ∎

Since we assume $N \in \mathcal{F}$, the grand coalition N is a closed coalition. Hence, efficiency of the ACT solution also follows from independence of closed coalitions, condition (i) of Definition 4.4. For a coalitional game, a closed coalition can be seen as a set of players whose performance is not affected by the other players of the game (condition (ii)) and also does not affect the performance of the other players (condition (iii)). Condition (ii) implies also that the ACT solution of the subgame obtained by the players of a closed coalition is the the same as the ACT solution for these players under the original game.

For the solution concept introduced in Aguilera et al. (2010), the property of independence of closed coalitions is not satisfied, because a closed coalition may not be a member of every maximal chain of the coalitional structure.

Example 4.2 Consider the coalitional structure $\mathcal{F} = \{\{1,2\},\{2,3\},\{1,2,3\},\{4,6\},\{5,7\},\{6,7\},\{1,2,3,4,5\},\{1,2,3,5,7,8\},N\}$ on $N = \{1,2,3,4,5,6,7,8\}$. $\mathcal{F}$ contains six maximal nested sets, $\mathcal{X}^1 = \{\{1,2\},\{1,2,3\},\{5,7\},\{1,2,3,5,7,8\}, N\}$, $\mathcal{X}^2 = \{\{2,3\},\{1,2,3\},\{5,7\},\{1,2,3,5,7,8\},N\}$, $\mathcal{X}^3 = \{\{1,2\},\{1,2,3\},\{6,7\},\{1,2,3,4,5\},N\}$, $\mathcal{X}^4 = \{\{2,3\},\{1,2,3\},\{6,7\},\{1,2,3,4,5\},N\}$, $\mathcal{X}^5 = \{\{1,2\},\{1,2,3\},\{4,6\},\{5,7\},N\}$, and $\mathcal{X}^6 = \{\{2,3\},\{1,2,3\},\{4,6\},\{5,7\},N\}$. The corresponding coalitional trees are depicted in Figure 3.

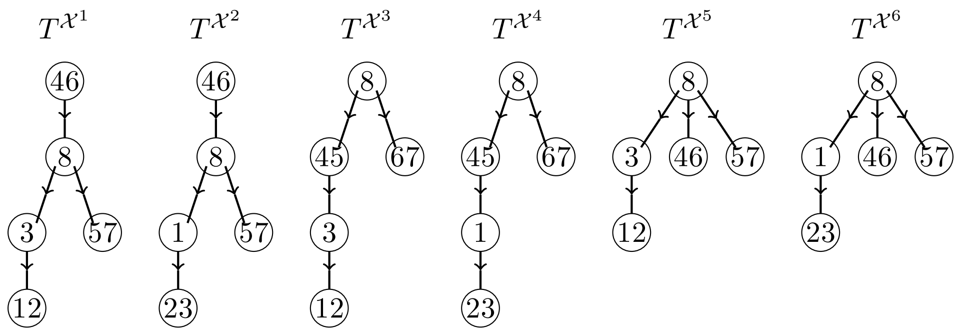


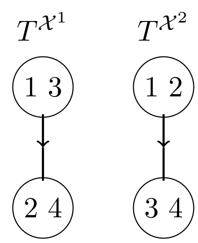
Figure 3: The coalitional trees of $\mathcal{F}$ in Example 4.2.

Coalition $\{1,2,3\}$ is a closed coalition and is in each coalitional tree a branch of the tree. For each of the six maximal nested sets the total payoff for the members of the coalition $\{1,2,3\}$ at the induced marginal vector is equal to its worth $v(\{1,2,3\})$. If for two coalitional games $(v,\mathcal{F})$ and $(w,\mathcal{F})$ it holds that $v(S) = w(S)$ for all feasible $S \subseteq \{1,2,3\}$, then at the ACT solution the payoffs for the players in $\{1,2,3\}$ are the same. Similarly, if $v(S) = w(S)$ holds for all feasible S satisfying $S \cap \{1,2,3\} = \emptyset$ or $S \supseteq \{1,2,3\}$, then at the ACT solution the payoffs for the players in $\{4,5,6,7,8\}$ are the same. Notice that $\{\{5,7\},\{1,2,3,5,7,8\},N\}$ is a maximal chain that does not contain the closed coalition $\{1,2,3\}$.

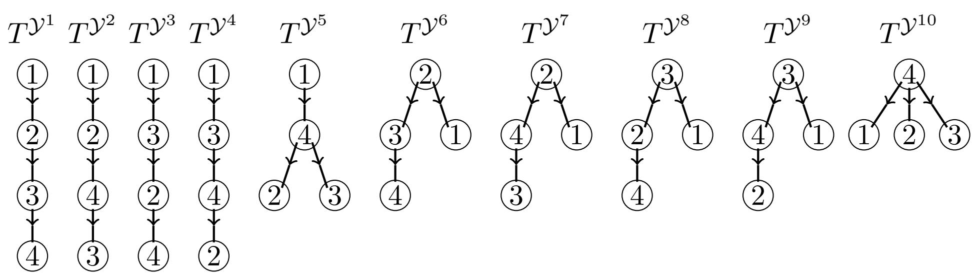
5 Special cases

Like for the ACT solution, in Koshevoy and Talman (2011) maximal nested sets are used as the main tool to define the GC solution for games with coalitional structures that are called building sets. A set system on N is a building set if all singleton coalitions and the union of all intersecting feasible coalitions are also feasible. Koshevoy and Talman (2011) also propose a way to use the GC solution for games with arbitrary coalitional structure by taking its building cover. Given a set system $\mathcal{F}$ on N, the building cover of $\mathcal{F}$, $\mathcal{B}(\mathcal{F})$, is defined as the smallest building set on N that contains $\mathcal{F}$. For a coalitional game $(v,\mathcal{F})$ on N, by using the Möbius inversion, they define the so called M-extension $v^{\mathcal{F}}$ of the characteristic function v and propose as GC solution of $(v,\mathcal{F})$ the GC solution of the game $(v^{\mathcal{F}},\mathcal{B}(\mathcal{F}))$. For a coalitional game $(v,\mathcal{F})$ where $\mathcal{F}$ is a building set, the ACT solution and GC solution coincide. For games with more general coalitional structure the ACT solution differs from the GC solution.

Example 5.1 Consider the coalitional game $(v, \mathcal{F})$ with coalitional structure $\mathcal{F} = \{\{2,4\},\{3,4\},\{1,2,3,4\}\}$ and $v(S) = |S|^2$ for all $S \in \mathcal{F}$. $\mathcal{F}$ has two maximal nested sets, $\mathcal{X}^1 = \{\{2,4\},\{1,2,3,4\}\}$ and $\mathcal{X}^2 = \{\{3,4\},\{1,2,3,4\}\}$, with corresponding coalitional trees depicted in Figure 4a. Since $\mathcal{F}$ contains no singleton coalitions and also not the union $\{2,3,4\}$ of the feasible coalitions $\{2,4\}$ and $\{3,4\}$, $\mathcal{F}$ is not a building set. The building cover of $\mathcal{F}$ is the collection $\mathcal{B}(\mathcal{F}) = \{\{1\},\{2\},\{3\},\{4\},\{2,4\},\{3,4\},\{2,3,4\}, \{1,2,3,4\}\}$, having ten maximal nested sets, $\mathcal{Y}^1 = \{\{4\},\{3,4\},\{2,3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^2 = \{\{3\},\{3,4\},\{2,3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^3 = \{\{4\},\{2,4\},\{2,3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^4 = \{\{2\},\{2,4\},\{2,3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^5 = \{\{2\},\{3\},\{2,3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^6 = \{\{1\},\{4\},\{3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^7 = \{\{1\},\{3\},\{3,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^8 = \{\{1\},\{4\},\{2,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^9 = \{\{1\},\{2\},\{2,4\},\{1,2,3,4\}\}$, $\mathcal{Y}^{10} = \{\{1\},\{2\},\{3\},\{1,2,3,4\}\}$, with corresponding coalitional trees depicted in Figure 4b.



a. The coalitional trees of $\mathcal{F}$.



b. The coalitional trees of $\mathcal{B}(\mathcal{F})$.

Figure 4: Example 5.1.

For the M-extension of v, $v^{\mathcal{F}}$, we obtain $v^{\mathcal{F}}(S) = v(S)$ if $S \in \mathcal{F}$, $v^{\mathcal{F}}(S) = 0$ if $|S| = 1$, and $v^{\mathcal{F}}(\{2,3,4\}) = v(\{2,4\}) + v(\{3,4\}) = 8$. It holds that $ACT(v, \mathcal{F}) = (6,4,4,2)^\top$ and $GC(v, \mathcal{F}) = GC(v^{\mathcal{F}}, \mathcal{B}(\mathcal{F})) = (4,4,4,4)^\top$. Notice that some of the marginal vectors that correspond to the coalitional trees of the building cover $\mathcal{B}(\mathcal{F})$ are the same.

In the next subsections we discuss some special cases of coalitional games. Since some of these cases, like complete coalitional structures and graphical coalitional structures, are also building sets, the ACT solution and GC solution give the same outcome for these cases.

5.1 Complete coalitional structures

A coalitional structure $\mathcal{F}$ on a set of players N is *complete* if $\mathcal{F} = 2^N$ where N is the set of all players. For a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$, if $\mathcal{F}$ is complete, all coalitions are feasible, which means that v is a TU game.

Lemma 5.1 *Given a complete coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S, Q \in \mathcal{X}$ it holds that $S \subseteq Q$ or $Q \subseteq S$.*

Proof. Suppose there exists a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and distinct $S, Q \in \mathcal{X}$ with $S \cap Q = \emptyset$. Since $\mathcal{F}$ is complete, it holds that $S \cup Q \in \mathcal{F}$. But this contradicts with condition (ii) of Definition 3.1. Since $\mathcal{X}$ is a maximal nested set and we can rule out the case $S \cap Q = \emptyset$ for distinct members S and Q, we end up with $S \subseteq Q$ or $Q \subseteq S$ for all $S, Q \in \mathcal{X}$. ∎

Lemma 5.2 *Given a complete coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S \in \mathcal{X}$ with $|S| \geq 2$, it holds that there exists $i \in S$ such that $S \setminus \{i\} \in \mathcal{X}$.*

Proof. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S \in \mathcal{X}$ with $|S| \geq 2$. Let $Q \in \mathcal{X}$ be a maximal proper subset of S in $\mathcal{X}$, i.e., either $Q = \emptyset$ or $Q \in \mathcal{X}$, $Q \subset S$ and there exists no $Q' \in \mathcal{X}$ with $S \supset Q' \supset Q$. Suppose $|S \setminus Q| \geq 2$. So there exists $i, j \in S \setminus Q$, $i \neq j$. Then $Q \subset Q \cup \{i\} \subset S$ and $Q \cup \{i\} \notin \mathcal{X}$. Since $\mathcal{F}$ is complete, $Q \cup \{i\} \in \mathcal{F}$, which contradicts that $\mathcal{X}$ is a maximal nested set of $\mathcal{F}$. ∎

Remark 5.1 For a complete coalitional structure $\mathcal{F}$ on N, every maximal nested set is a maximal chain and $|\overline{\mathcal{X}}^{\mathcal{F}}| = n!$.

Remark 5.1 is an immediate result of Lemma 5.1 and Lemma 5.2. To form a maximal nested set, first we can include the grand coalition. By Lemma 5.2 we should also include a subset of the grand coalition with cardinality one less. The number of possibilities is n for this step. At the next step we will have $n-1$ possibilities and so on. Hence, in total there are $n!$ possibilities, each of which corresponds to a maximal nested set being also a maximal chain.

For TU games, the Shapley value is the best known single-valued solution concept, see Shapley (1953). To find the Shapley value, all permutations over the players are considered and a marginal vector is associated to each permutation, at which each player receives his marginal contribution when joining to his set of successors in the permutation. The Shapley value is the average of all these marginal vectors. Clearly, these permutations correspond one-to-one to the set of maximal nested sets. Let $Sh(v)$ stand for the Shapley value of a TU game v.

Corollary 5.1 *For a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ with complete coalitional structure $\mathcal{F}$ on N, it holds that $ACT(v, \mathcal{F}) = Sh(v)$.*

5.2 Graphical coalitional structures

A *graph* on N consists of a set of nodes, being the elements of N, and a collection of unordered pairs of nodes $L \subseteq L_N^c$, where $L_N^c = \{\{i,j\} \,|\, i,j \in N,\ i \neq j\}$ is the complete graph without loops on N and an unordered pair $\{i,j\} \in L$ is a *link* between i and j. For a graph L on N, a sequence of different nodes $(i_1, \ldots, i_k)$, $k \geq 2$, is a *path* in L between node i_1 and node i_k if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. A coalition $S \in 2^N$ is a *network* if S is *connected*, i.e., for any two different players $i, j \in S$ there is a path in L of nodes in S between i and j. For $S \in 2^N$, a network

$Q \subseteq S$ is a *component* of S in L if Q cannot form a larger network with any $i \in S \setminus Q$. For $S \in 2^N$, $K^L(S)$ denotes the collection of networks in L that are subsets of S and $\widehat{K}^L(S)$ denotes the collection of components of S in L.

A coalitional structure $\mathcal{F}$ on N is *graphical*, if $\mathcal{F}$ is the collection of all connected sets of players of a graph L on N, i.e., $\mathcal{F} = K^L(N)$. In order to have the grand coalition N as a feasible set, throughout this subsection, we assume that N forms a network in the graph that induces the coalitional structure. In case the set of feasible coalitions is restricted to the collection of sets of connected players of an undirected graph defined on N, the coalitional game is called a graph game, see Myerson (1977).

Lemma 5.3 *Given a graphical coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ it holds that $P_{\mathcal{X}}(i) = \{i\}$ for all $i \in N$.*

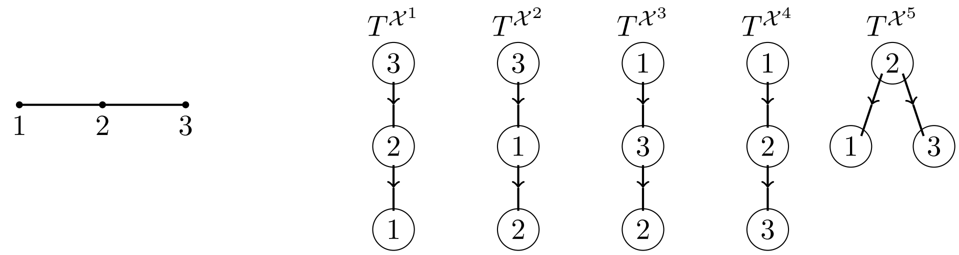
Proof. Let the graphical coalitional structure $\mathcal{F}$ on N be induced by a graph L on N and take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Suppose there exists $i \in N$ for which $j \in P_{\mathcal{X}}(i)$, for some $j \neq i$. Then $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$. Let $K \in \widehat{K}^L(M_{\mathcal{X}}(i) \setminus \{i\})$ be the component of $M_{\mathcal{X}}(i) \setminus \{i\}$ in L containing j. Then $K \in \mathcal{F}$ and $K \notin \mathcal{X}$. Since L is a graph on N, $\mathcal{X} \cup \{K\}$ is a nested set, contradicting that $\mathcal{X}$ is a maximal nested set. ∎

An immediate result of Lemma 5.3 is that, if the coalitional structure of a coalitional game is graphical then all coalitional trees corresponding to the maximal nested sets are trees.

Lemma 5.4 *Given a graphical coalitional structure $\mathcal{F}$ on N induced by a graph L on N, for any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ it holds that $\overline{S}_{T^{\mathcal{X}}}(P') \in \widehat{K}^L(S_{T^{\mathcal{X}}}(P))$ if $(P, P') \in T^{\mathcal{X}}$.*

Proof. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and let $(P, P') \in T^{\mathcal{X}}$. As a direct result of Theorem 3.1 and since $\mathcal{F} = K^L(N)$, we have $\overline{S}_{T^{\mathcal{X}}}(P) \in K^L(N)$ and $\overline{S}_{T^{\mathcal{X}}}(P') \in K^L(N)$. Let $S_1 \ldots, S_k$ be the components of $S_{T^{\mathcal{X}}}(P)$ in L, i.e., $S_i \in \widehat{K}^L(S_{T^{\mathcal{X}}}(P))$ holds for all $i = 1, \ldots, k$. First, suppose $\overline{S}_{T^{\mathcal{X}}}(P') \subset S_h$ for some $h \in \{1, \ldots, k\}$. Since $(P, P') \in T^{\mathcal{X}}$, we have $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(P')$ and there exists no $X \in \mathcal{X}$ such that $M_{\mathcal{X}}(P) \supset X \supset M_{\mathcal{X}}(P')$. Hence, $S_h \notin \mathcal{X}$, which contradicts that $\mathcal{X}$ is a maximal nested set. Next, suppose $\overline{S}_{T^{\mathcal{X}}}(P') \cap S_h \neq \emptyset$ and $\overline{S}_{T^{\mathcal{X}}}(P') \cap S_m \neq \emptyset$ for some $h, m \in \{1, \ldots, k\}$, $h \neq m$. Since $\overline{S}_{T^{\mathcal{X}}}(P') \in K^L(N)$, this contradicts that S_h and S_m both are components of $S_{T^{\mathcal{X}}}(P)$ in L. Hence, $\overline{S}_{T^{\mathcal{X}}}(P')$ is equal to S_h for some $h \in \{1, \ldots, k\}$, which completes the proof. ∎

Example 5.2 Consider the graph $L = \{\{1, 2\}, \{2, 3\}\}$ on $\{1, 2, 3\}$ as depicted in Figure 5a. $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ is the corresponding graphical coalitional structure on N. $\mathcal{F}$ has five maximal nested sets, $\mathcal{X}^1 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, $\mathcal{X}^2 = \{\{2\}, \{1, 2\}, \{1, 2, 3\}\}$, $\mathcal{X}^3 = \{\{2\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathcal{X}^4 = \{\{3\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathcal{X}^5 = \{\{3\}, \{1\}, \{1, 2, 3\}\}$, with corresponding coalitional trees as depicted in Figure 5b.



a. The graph L.

b. The coalitional trees of $\mathcal{F}$.

Figure 5: Example 5.2.

Since the collection of networks of a connected graph is a building set, it holds that on the class of graph games the GC solution and the average coalitional tree solution coincide.

5.3 Majoritarian coalitional structures

In voting theory literature, a quota rule refers to the case where in order to accept a proposition the number of supporting votes must be greater than or equal to a predetermined threshold number which is generally greater than or equal to half of the total number of votes. Given the set of players N as the set of voters and some real number q, $\frac{1}{2} \leq q \leq 1$, let $\mathcal{F}^q = \{S \in 2^N \mid |S| \geq qn\}$ be the set of winning coalitions under the quota majority rule with quota q. Referring to the example of political parties in the introduction where not all coalitions are feasible, in this subsection we consider feasible winning coalitions where both the size of the coalition should exceed a predetermined quota and the coalition itself should be feasible. A coalitional structure $\mathcal{F}$ on N, is *q-majoritarian* if $\mathcal{F} \subseteq \mathcal{F}^q$. Hence a q-majoritarian coalitional structure on N contains some or all coalitions with cardinality greater than or equal to qn.

Remark 5.2 For a q-majoritarian coalitional structure $\mathcal{F} \subseteq \mathcal{F}^q$ on N with $\frac{1}{2} \leq q \leq 1$, $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and S, $S' \in \mathcal{X}$, it holds that $S \subseteq S'$ or $S' \subseteq S$.

A direct result of Remark 5.2 is that all maximal nested sets of a q-majoritarian coalitional structure are maximal chains as in Aguilera et al. (2010). Hence, for this particular case, the ACT solution is the same as the solution defined in Aguilera et al. (2010). Additional to the restriction on the set system, assume that the worth of all feasible coalitions is equal to 1. In this case we may focus only on minimal feasible winning coalitions.

Given a q-majoritarian coalitional structure $\mathcal{F}$ on N, let $\mathcal{M}^{\mathcal{F}}$ be the collection of minimal feasible winning coalitions. Note that for a q-majoritarian coalitional structure $\mathcal{F}$, if $N \in \mathcal{M}^{\mathcal{F}}$ then there exist no other member of $\mathcal{M}^{\mathcal{F}}$. This can only happen in case of unanimity voting when $\mathcal{F} = \{N\}$. Among the members of $\mathcal{M}^{\mathcal{F}}$ let $\mathcal{M}^{\mathcal{F}}(i)$ be the collection of minimal feasible winning coalitions that contain player i, $i \in N$. For some $i \in N$ it can be the case that $\mathcal{M}^{\mathcal{F}}(i) = \emptyset$ and in case of unanimity voting when $\mathcal{F} = \{N\}$ we have $\mathcal{M}^{\mathcal{F}}(i) = \{N\}$ for all $i \in N$.

Corollary 5.2 *For a coalitional game $(v, \mathcal{F})$ where $\mathcal{F} \subset \mathcal{F}^q$ is q-majoritarian for* *some* q, $\frac{1}{2} \leq q \leq 1$, *and* $v(S) = 1$ *for all* $S \in \mathcal{F}$, *it holds that*

$$ACT_i(v, \mathcal{F}) = \frac{1}{|\mathcal{M}^{\mathcal{F}}|} \sum_{S \in \mathcal{M}^{\mathcal{F}}(i)} \frac{1}{|S|} \ , i \in N.$$

5.4 Individualist coalitional structures

A coalitional structure $\mathcal{F}$ is *individualist* if it contains all singletons, i.e., $\{i\} \in \mathcal{F}$ holds for all $i \in N$. In case the coalitional structure $\mathcal{F}$ is an arbitrary collection of feasible coalitions, for some $S \in 2^N$ there may not exist disjoint feasible coalitions whose union is equal to S. On the other hand, if $\mathcal{F}$ is individualist, then for any nonempty $S \in 2^N$ there exists a partition of S in terms of feasible coalitions. Notice that a building set is an individualist coalitional structure.

Lemma 5.5 *Given an individualist coalitional structure* $\mathcal{F}$ *on* N, *for any* $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ *it holds that* $P_{\mathcal{X}}(i) = \{i\}$ *for all* $i \in N$.

Proof. Suppose there exists $\mathcal{X} \in \mathcal{X}^{\mathcal{F}}$ and $i \in N$ such that $j \in P_{\mathcal{X}}(i)$ for some $j \neq i$. Hence, $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$ and since $\{i\}, \{j\} \in \mathcal{F}$ there exists $S \in \mathcal{X}$ such that $S \subset M_{\mathcal{X}}(i)$ and for all $S' \supset S$, $S' \not\supseteq M_{\mathcal{X}}(i)$. This implies that $\mathcal{X} \cup \{\{i\}\}$ or $\mathcal{X} \cup \{\{j\}\}$ is a nested set, which contradicts that $\mathcal{X}$ is a maximal nested set. ∎

Similar to the special case where the coalitional structure is a building set, a direct result of this lemma is that all coalitional trees induced by maximal nested sets of an individualist coalitional structure are trees. As the following example illustrates an individualist coalitional structure is not necessarily graphical or a building set.

Example 5.3 Consider an individualist coalitional structure $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3, 4\}\}$ on $\{1, 2, 3, 4\}$. There are eight maximal nested sets, $\mathcal{X}^1 = \{\{3\}, \{2, 3\}, \{4\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^2 = \{\{2\}, \{2, 3\}, \{4\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^3 = \{\{1\}, \{1, 2\}, \{4\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^4 = \{\{2\}, \{1, 2\}, \{4\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^5 = \{\{2\}, \{2, 3\}, \{1\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^6 = \{\{3\}, \{2, 3\}, \{1\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^7 = \{\{2\}, \{1, 2\}, \{3\}, \{1, 2, 3, 4\}\}$, $\mathcal{X}^8 = \{\{1\}, \{1, 2\}, \{3\}, \{1, 2, 3, 4\}\}$. The graphical representation of the corresponding coalitional trees is depicted in Figure 4.

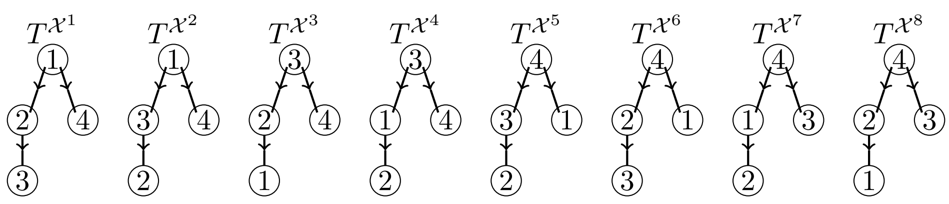


Figure 6: The coalitional trees of $\mathcal{F}$ in Example 5.3.

5.5 Partitional coalitional structures

In this subsection we consider coalitional structures where for each player the only alternative of not participating in the grand coalition is to participate in a unique smaller coalition. A coalitional structure $\mathcal{F}$ is *partitional* if it contains the grand coalition N and a proper partition of N.

Lemma 5.6 *If $\mathcal{F}$ is a partitional coalitional structure on N, then $|\overline{\mathcal{X}}^{\mathcal{F}}| = |\mathcal{F}| - 1$.*

Proof. Let $\mathcal{F} = \{S_1, \dots, S_k, N\}$ where $S_1, \dots, S_k$ forms a partition of N for some $k \geq 2$. Then $\mathcal{X}$ is a maximal nested set of $\mathcal{F}$ if and only if there exists $i \in \{1, \dots, k\}$ such that $\mathcal{X} = \mathcal{F} \setminus \{S_i\}$. So $|\overline{\mathcal{X}}^{\mathcal{F}}| = k$, which completes the proof. ∎

A direct result of this lemma is that, given a partitional coalitional structure, each collection of feasible coalitions that excludes only one of the partition members is a maximal nested set.

Lemma 5.7 *Given a partitional coalitional structure $\mathcal{F} = \{S_1, \dots, S_k, N\}$, if $i \in S_h$ for some $h \in \{1, \dots, k\}$, then $P_{\mathcal{X}}(i) = S_h$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$.*

Proof. Let $\mathcal{F} = \{S_1, \dots, S_k, N\}$ be a partitional coalitional structure and $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Take any $h \in \{1, \dots, k\}$ and $i \in S_h$. Suppose $S_h \in \mathcal{X}$. Since $S_1, \dots, S_k$ forms a partition of N and $S_h \in \mathcal{X}$, we have $M_{\mathcal{X}}(j) = S_h$ for all $j \in S_h$. So, $P_{\mathcal{X}}(i) = S_h$. Next, suppose $S_h \notin \mathcal{X}$. Since $S_1, \dots, S_k$ forms a partition of N and $S_h \notin \mathcal{X}$, we have $M_{\mathcal{X}}(j) = N$ for all $j \in S_h$. Hence, again $P_{\mathcal{X}}(i) = S_h$, which completes the proof. ∎

Since coalitional trees are defined on the sets of equivalent players, by Lemma 5.7, for a partitional coalitional structure $\mathcal{F} = \{S_1, \dots, S_k, N\}$, all of the induced coalitional trees are defined on the partition $\{S_1, \dots, S_k\}$.

Theorem 5.1 *Given a coalitional game $(v, \mathcal{F}) \in \mathcal{G}_N$ with partitional coalitional structure $\mathcal{F} = \{S_1, \dots, S_k, N\}$, for all $m \in \{1, \dots, k\}$ it holds that*

$$\sum_{i \in S_m} ACT_i(v, \mathcal{F}) = v(S_m) + \frac{1}{k}(v(N) - \sum_{h=1}^{k} v(S_h)).$$

Proof. Take any $m \in \{1, \dots, k\}$. By Lemma 5.6 we have $|\overline{\mathcal{X}}^{\mathcal{F}}| = k$. Let $\overline{\mathcal{X}}^{\mathcal{F}} = \{\mathcal{X}^1, \dots \mathcal{X}^k\}$. Then $k-1$ of these maximal nested sets contain S_m and only one of them does not contain S_m. Without loss of generality let $S_m \in \mathcal{X}^h$ holds for $h = 1, \dots, k-1$. Then $\sum_{i \in S_m} m_i^{\mathcal{X}^h}(v, \mathcal{F}) = v(S_m)$ for $h \in \{1, \dots, k-1\}$ and for $\mathcal{X}^k$ we have $\sum_{i \in S_m} m_i^{\mathcal{X}^k}(v, \mathcal{F}) = v(N) - \sum_{h=1}^{k-1} v(S_h)$. Since the ACT solution is the average of these marginal vectors, we obtain $\sum_{i \in S_m} ACT_i(v, \mathcal{F}) = v(S_m) + (v(N) - \sum_{h=1}^{k} v(S_h))/k$. ∎

As Theorem 5.1 shows, given a coalitional game with a partitional coalitional structure, each member of the partition receives its worth plus an equal share of the total contribution of all members of the partition while forming the grand coalition. On the individual level, each player receives an equal share of the total payoff available to the partition member he belongs to. This is confirmed by the property of equal treatment of equivalent players, because for the underlying game the players in any partition member are equivalent to each other.

Example 5.4 Consider a partitional coalitional structure $\mathcal{F} = \{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$. $\mathcal{F}$ has four maximal nested sets, $\mathcal{X}^1 = \{\{4, 5\}, \{6\}, \{7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$, $\mathcal{X}^2 = \{\{1, 2, 3\}, \{6\}, \{7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$, $\mathcal{X}^3 = \{\{1, 2, 3\}, \{4, 5\}, \{7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$ and $\mathcal{X}^4 = \{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{1, 2, 3, 4, 5, 6, 7\}\}$, with corresponding coalitional trees as depicted in Figure 7.

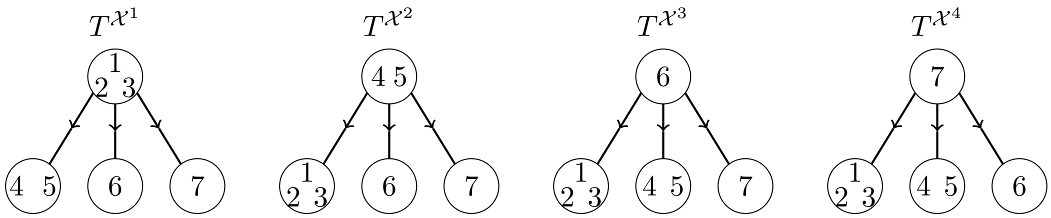


Figure 7: The coalitional trees of $\mathcal{F}$ in Example 5.4.

5.6 (Weakly) union closed coalitional structures

In their paper Algaba et al. (2003) consider antimatroids as a coalitional structure for a cooperative games. Antimatroids are set systems that are also closed under union. A coalitional structure $\mathcal{F}$ is *union closed* if $S \cup Q \in \mathcal{F}$ whenever S, $Q \in \mathcal{F}$. Another concept of set systems is augmenting systems, introduced by Bilbao (2003). Augmenting systems are set systems that are also weakly union. A coalitional structure $\mathcal{F}$ is *weakly union closed* if $S \cup Q \in \mathcal{F}$ whenever S, $Q \in \mathcal{F}$ and $S \cap Q \neq \emptyset$.

Lemma 5.8 *Given a union closed coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^F$, if X_1, $X_2 \in \mathcal{X}$ then $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$*

Proof. Suppose there exist X_1, $X_2 \in \mathcal{X}$ where $X_1 \nsubseteq X_2$ and $X_2 \nsubseteq X_1$. Since $\mathcal{F}$ is closed under union, $X_1 \cup X_2 \in \mathcal{F}$ which violates condition (ii) of Definition 3.1. ∎

The lemma implies that in case of union closed coalitional structure all maximal nested sets are maximal chains and therefore, for this particular case, the ACT solution is the same as the solution defined in Aguilera et al. (2010).

Lemma 5.9 *Given a weakly union closed coalitional structure $\mathcal{F}$ on N, for any $\mathcal{X} \in \overline{\mathcal{X}}^F$ and $P \in \mathcal{P}^{\mathcal{X}}$ it holds that $S_{T^{\mathcal{X}}}(P)$ has a unique maximal partition into feasible coalitions of $\mathcal{F}$.*

Proof. Take any $\mathcal{X} \in \overline{\mathcal{X}}^F$ and $P \in \mathcal{P}^{\mathcal{X}}$. By condition (iii) of Theorem 3.1, $S_{T^{\mathcal{X}}}(P)$ has a unique maximal partition into elements of $\mathcal{X}$, where $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid (P, P') \in T^{\mathcal{X}}\}$ is that partition. Since $\mathcal{X}$ is a maximal nested set, $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid (P, P') \in T^{\mathcal{X}}\}$ is also a maximal partition of $S_{T^{\mathcal{X}}}(P)$ into feasible coalitions. Let $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid (P, P') \in T^{\mathcal{X}}\} = \{S_1, \dots, S_k\}$ and suppose there exists another maximal partition $\{Q_1, \dots, Q_m\}$ of $S_{T^{\mathcal{X}}}(P)$ into feasible coalitions. Then there exists $h \in \{1, \dots, k\}$ and $\ell \in \{1, \dots, m\}$ such that $S_h \nsubseteq Q_\ell$, $Q_\ell \nsubseteq S_h$ and $S_h \cap Q_\ell \neq \emptyset$. Since $\mathcal{F}$ is weakly union closed, the union of all S_j, $j \in \{1, \dots, k\}$, such that $S_j \cap Q_\ell \neq \emptyset$ is a feasible coalition, which contradicts that $\mathcal{X}$ is a maximal nested set. ∎

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